

# ON FUNCTIONS WITHOUT A NORMAL ORDER

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ABSTRACT. The method of Turán in establishing the normal order for the number of prime divisors of a number is used to show that a certain class of arithmetic functions do not have a normal order.

## 1. INTRODUCTION

The normal order of an arithmetic function, defined in [2, p. 356], measures the ‘usual size’ of the function: A function  $\psi(n) \geq 0$  is said to have a normal order  $f(n)$  if, to every  $\epsilon > 0$ , the number of  $n \leq x$  for which  $|\psi(n) - f(n)| < \epsilon f(n)$  is  $o(x)$ , as  $x \rightarrow \infty$ . It is tacitly assumed that  $f(n)$  is increasing—otherwise, every such  $\psi(n)$  has itself as normal order.

The notion was first introduced by G. H. Hardy and S. Ramanujan [1], who proved that  $\omega(n)$ , the number of distinct prime divisors of  $n$ , has the normal order  $\log \log n$ . Their proof was much simplified by P. Turán ([2, p. 356], [5]), who showed that the result can be established from the asymptotic formulae for the first and the second moments of  $\omega(n)$ ; indeed it is sometimes said that probabilistic number theory stems from [5]. By applying Turán’s method ‘in reverse’, so to speak, S. L. Segal [4] showed that Euler’s totient function  $\phi(n)$  does not have a normal order. We distil the argument used by Segal, thereby extending his result to a certain class of arithmetic functions.

## 2. A CLASS OF FUNCTIONS WITHOUT A NORMAL ORDER

Let  $\mathcal{M}$  denote the class of arithmetic functions  $\psi$  for which there are positive constants  $A, B, C$  such that  $0 \leq \psi(n) < Cn$  and, as  $x \rightarrow \infty$ ,

$$(1) \quad \sum_{n \leq x} \psi(n) \sim \frac{Ax^2}{2} \quad \text{and} \quad \sum_{n \leq x} \psi^2(n) \sim \frac{Bx^3}{3}.$$

**Theorem.** *Let  $\psi \in \mathcal{M}$ . If  $A^2 < B$  then  $\psi$  does not have a normal order.*

*Proof.* Let  $A, B, C$  be constants associated with  $\psi \in \mathcal{M}$ , and set

$$(2) \quad R(x) = \sum_{n \leq x} (\psi(n) - An) = o(x^2), \quad \text{as } x \rightarrow \infty.$$

Suppose that  $\psi(n)$  has the normal order  $f(n)$ ; we may assume without loss that  $f(n) < 2Cn$ , so that  $|\psi(n) - f(n)| \leq \max\{\psi(n), f(n)\} < 2Cn$ . Making use of (2),

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and  $f(n)$  being increasing, we find, by partial summation, that

$$(3) \quad \left| \sum_{n \leq x} (\psi(n) - An) f(n) \right| \leq \max_{n \leq x} |R(n)| \left\{ \sum_{n \leq x-1} (f(n+1) - f(n)) + f(x) \right\} \\ = o(x^3) \quad \text{as } x \rightarrow \infty.$$

Let  $\epsilon > 0$ . Appealing to the definition of normal order and separating terms depending on whether  $|\psi(n) - f(n)| < \epsilon f(n)$ , or not, we then have, as  $x \rightarrow \infty$ ,

$$(4) \quad \sum_{n \leq x} (\psi(n) - f(n))^2 \leq 4\epsilon^2 C^2 \sum_{n \leq x} n^2 + 4C^2 x^2 o(x) = \frac{4\epsilon^2 C^2 x^3}{3} + o(x^3).$$

From (1), (3), (4), together with

$$\begin{aligned} \psi^2(n) &= A^2 n^2 + (\psi(n) - f(n))^2 + 2(\psi(n) - An)f(n) - (f(n) - An)^2 \\ &\leq A^2 n^2 + (\psi(n) - f(n))^2 + 2(\psi(n) - An)f(n), \end{aligned}$$

we now have, on summing over  $n \leq x$ ,

$$\frac{Bx^3}{3} + o(x^3) \leq \frac{A^2 x^3}{3} + \frac{4\epsilon^2 C^2 x^3}{3} + o(x^3).$$

If  $\epsilon = \epsilon(A, B, C)$  is sufficiently small, and  $x$  is large, then the inequality here is untenable for  $A^2 < B$ . The theorem is proved.

### 3. SEGAL'S THEOREM ON $\phi(n)$

**Lemma.** *For Euler's function  $\phi(n)$ , we have, as  $x \rightarrow \infty$ ,*

$$(5) \quad \sum_{n \leq x} \phi(n) = \frac{Ax^2}{2} + O(x \log x)$$

and

$$(6) \quad \sum_{n \leq x} \phi^2(n) = \frac{Bx^3}{3} + O(x^2 \log^2 x),$$

where, for primes  $p$ ,

$$A = \prod_p \left(1 - \frac{1}{p^2}\right) \quad \text{and} \quad B = \prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right).$$

Thus  $\phi \in \mathcal{M}$ , and it is readily seen that  $A^2 < B$ , so that  $\phi(n)$  does not have a normal order. The asymptotic formula (5) is due to F. Mertens [3], and (6) is due to Segal [4], who gave a somewhat elaborate proof. For completeness sake, we give the proof of the lemma here.

*Proof.* By Möbius inversion, we have

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d},$$

where  $\mu(n)$  is the Möbius function; the formula can also be verified by taking  $n$  to be a prime power, and noting that the functions involved are multiplicative. It

follows that, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \sum_{n \leq x} \phi(n) &= \sum_{ab \leq x} a\mu(b) = \sum_{b \leq x} \mu(b) \sum_{a \leq x/b} a = \sum_{b \leq x} \mu(b) \left\{ \frac{1}{2} \left( \frac{x}{b} \right)^2 + O\left( \frac{x}{b} \right) \right\} \\ &= \frac{Ax^2}{2} + E_1(x) + E_2(x), \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{b=1}^{\infty} \frac{\mu(b)}{b^2} = \prod_p \left( 1 - \frac{1}{p^2} \right), \\ E_1(x) &= O\left( x^2 \sum_{b > x} \frac{1}{b^2} \right) = O(x), \quad E_2(x) = O\left( x \sum_{b \leq x} \frac{1}{b} \right) = O(x \log x), \end{aligned}$$

so that (5) is proved.

Again, from the functions involved being multiplicative, it can be checked that

$$\left( \sum_{d|n} \frac{\mu(d)}{d} \right)^2 = \sum_{a|n} \frac{\mu^2(a)}{a^2} g(a), \quad \text{where } g(a) = \prod_{p|a} (1 - 2p).$$

Thus, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \sum_{n \leq x} \phi^2(n) &= \sum_{ab \leq x} a^2 \mu^2(b) g(b) = \sum_{b \leq x} \mu^2(b) g(b) \left\{ \frac{x^3}{3b^3} + O\left( \frac{x^2}{b^2} \right) \right\} \\ &= \frac{Bx^3}{3} + E_3(x) + E_4(x) \end{aligned}$$

where

$$\begin{aligned} B &= \sum_{b=1}^{\infty} \frac{\mu^2(b) g(b)}{b^3} = \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right), \\ E_3(x) &= O\left( x^3 \sum_{b > x} \frac{|g(b)|}{b^3} \right), \quad E_4(x) = O\left( x^2 \sum_{b \leq x} \frac{|g(b)|}{b^2} \right). \end{aligned}$$

Apply the bound  $|g(b)| \leq \prod_{p|b} (2p) \leq 2^{\omega(b)} b \leq d(b)b$ , where  $d(n)$  is the divisor function, and consider

$$\begin{aligned} \sum_{b > x} \frac{d(b)}{b^2} &= \sum_{uv > x} \frac{1}{u^2 v^2} = \sum_{u \leq x} \frac{1}{u^2} \sum_{v > x/u} \frac{1}{v^2} + \sum_{u > x} \frac{1}{u^2} \sum_{v=1}^{\infty} \frac{1}{v^2} \\ &= O\left( \frac{1}{x} \sum_{u \leq x} \frac{1}{u} \right) + O\left( \sum_{u > x} \frac{1}{u^2} \right) = O\left( \frac{\log x}{x} \right), \\ \sum_{b \leq x} \frac{d(b)}{b} &= \sum_{uv \leq x} \frac{1}{uv} = O(\log^2 x). \end{aligned}$$

Thus  $E_3(x) = O(x^2 \log x)$  and  $E_4(x) = O(x^2 \log^2 x)$ , and the lemma is proved.

Finally, we remark that Turán's method is more flexible than what is used to establish the theorem. Roughly speaking, the argument applies to any  $\psi(n)$  for which the second moment sum  $\sum_{n \leq x} \psi^2(n)$  is substantially larger than what 'might

be expected' from the bound for the first moment sum  $\sum_{n \leq x} \psi(n)$ . For example, from

$$\sum_{n \leq x} d(n) \sim x \log x \quad \text{and} \quad \sum_{n \leq x} d^2(n) \sim \frac{x \log^3 x}{\pi^2}, \quad \text{as } x \rightarrow \infty,$$

we see that the average value for  $d(n)$  is  $\log n$ , whereas the average value for  $d^2(n)$  is  $\log^3 n / \pi^2$ , which is significantly larger than  $\log^2 n$ . The proof of the theorem can easily be adapted to show that  $d(n)$  does not have a normal order.

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